Generic Torelli theorem for hypersurfaces of Kahler C-spaces with $b_2 = 1$.

KONNO, Kazuhiro

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Kazuhiro KONNO (今野 一宏. 東北大.理)

0. Introduction. A simply connected homogeneous Kähler manifold is called a Kähler C-space. A Kähler C-space is known to be rational and admit an "algebraic cell decomposition". In this report, we review our recent result [Kon.2] on the generic Torelli problem of smooth hypersurfaces of a Kähler C-space. We shall omit most proofs because of the limit on pages.

Let $\mathcal{M}$ be the coarse moduli space of varieties under consideration and $P : \mathcal{M} \longrightarrow \Gamma\backslash\mathcal{D}$ be the period map. We say that the *generic Torelli* theorem holds if P is generically injective. The theory of the *infinitesimal variation of Hodge structure* (abbreviated IVHS) opened a way to attack the generic Torelli problem (especially for hypersurfaces). In fact, using IVHS, Donagi [D] showed the generic Torelli theorem for *projective* hypersurfaces of almost all degree.

Recall that a Kähler C-space with $b_2 = 1$ can be constructed by a possible pair $(\mathfrak{g}, \alpha_r)$ of a complex simple Lie algebra $\mathfrak{g}$ and a simple root α_r of $\mathfrak{g}$. We define two positive integers $k(Y)$ and $m(Y)$ for each $Y = (\mathfrak{g}, \alpha_r)$:

(0.1) *The canonical line bundle* K_Y *is expressed as* $K_Y = \mathcal{O}_Y(-k(Y))$, *where* $\mathcal{O}_Y(-k(Y))$ *denotes the* $-k(Y)$*-th power of the ample generator* $\mathcal{O}_Y(1)$ *of* $Pic(Y)$ (cf. [Kon.1, (1.2)]).

(0.2) $m(Y) = n_r(\alpha_0) + 1$, *where* $n_r(\alpha_0)$ *denotes the coefficient of* α_r *in the highest root* α_0 *of* $\mathfrak{g}$ *which has the same length as* α_r (cf. [H, p. 66, Table 2]).

In [Kon.2], we have shown the following along the same line as in [D] and [G].

(0.3) **Generic Torelli Theorem.** *Let* Y *be a Kähler C-space with* $b_2(Y) = 1$ *which is not isomorphic to a projective space. Assume that a positive integer* d *satisfies the conditions,*

(1) $d > m(Y)$, *and* (2) $G.C.D.(d, k(Y)) < d/2$.

If $\mathcal{M}_d$ *denotes the coarse moduli space of smooth hypersurfaces of degree* d *of* Y , *then the period map* $P : \mathcal{M}_d \longrightarrow \Gamma\backslash\mathcal{D}$ *is generically injective except in the following cases :*

(a) $Y = (A_4, \alpha_2)$: $d = 3$,

(b) $Y = Q^N$: $d|N \pm 1$, $d|N \pm 2$, $d|N-4$,

(c) $Y = (C_l, \alpha_2)$, $l \geq 3$: $d|k(Y) \pm 1$, $d|k(Y) - 2$,

(d) $Y = (D_l, \alpha_l)$, $l = 5, 6$: $d = 3$,

(e) $Y = (B_l, \alpha_2)$; $l \geq 3$, (D_l, α_2); $l \geq 4$, (E_6, α_2) , (E_7, α_1) , (E_8, α_8) , (F_4, α_1) : $d|k(Y) - 1$, $d|k(Y) - 2$,

(f) $Y = (B_l, \alpha_3)$; $l \geq 4$, (B_l, α_4); $l \geq 5$, (C_l, α_3); $l \geq 4$, (C_l, α_4); $l \geq 5$, (D_l, α_3); $l \geq 5$, (D_l, α_4); $l \geq 6$, (E_6, α_4) , (E_7, α_3) , (E_7, α_6) , (E_8, α_1) , (F_4, α_3) : $d|k(Y) - 1$,

(g) $Y = (F_4, \alpha_4)$: $d|k(Y) \pm 1$, $d|k(Y) - 2$, $d|k(Y) - 3$.

It is known (cf. [Ki.2] and [Sak]) that the Kähler C-space (C_l, α_2) , $l \geq 3$, (resp. (F_4, α_4)) is a hypersurface of degree

1 of the grassmannian $Grass(2, 2l) = (A_{2l-1}, \alpha_2)$ (resp. the symmetric space of type EIII $= (E_6, \alpha_1)$). Of course, a complex quadric Q^N is a hypersurface of degree 2 of $\mathbb{P}^{N+1}$. Thus, our result contains informations about some *complete intersections* though the generic Torelli still remains open even for ones in a projective space.

In case of hypersurfaces of irreducible Hermitian symmetric spaces of compact type, (0.3) is obtained by M.-H. Saito [Sai], independently.

1. Kähler C-spaces with $b_2 = 1$.

(1.1) Let $\mathfrak{g}$ be a complex simple Lie algebra. If $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$ and

$$\Delta = \{\alpha_1, \cdots, \alpha_l\} , \quad l = \text{rank } \mathfrak{g} ,$$

is a base of the root system Φ of $\mathfrak{g}$ with respect to $\mathfrak{h}$, we denote by Φ^+ (resp. Φ^-) the set of all positive (resp. negative) roots. Then we have a *Cartan decomposition*

$$\mathfrak{g} = \mathfrak{h} + \sum_{\alpha\in\Phi^-} \mathfrak{g}_\alpha + \sum_{\alpha\in\Phi^+} \mathfrak{g}_\alpha .$$

Let $\{\lambda_1, \cdots, \lambda_l\}$ be the fundamental weight system with respect to Δ , that is, $\langle\lambda_i, \alpha_j\rangle := 2(\lambda_i, \alpha_j)/(\alpha_j, \alpha_j) = \delta_{ij}$, where $(\ , \ )$ denotes the Euclidean scalar product induced by the Killing form on the real vector space spanned by Φ in $\mathfrak{h}^*$.

(1.2) Choose a simple root α_r, $1 \le r \le l$, and put

$$\Phi(u) = \{ \alpha \in \Phi \mid n_r(\alpha) \ge 0 \} ,$$

where $n_r(\alpha)$ is the coefficient of α_r when we express α as an integral sum of α_i's : $\alpha = \sum n_i(\alpha)\alpha_i$. Put

$$\mathfrak{u} = \mathfrak{h} + \sum_{\alpha\in\Phi(\mathfrak{u})} \mathfrak{g}_\alpha .$$

Take a simply connected complex simple Lie group G and a connected Lie subgroup U of G in such a way that Lie $G = \mathfrak{g}$ and Lie $U = \mathfrak{u}$. Then the factor space $Y = G/U$ is a Kähler C-space and its second Betti number is one. Conversely, every Kähler C-space with $b_2 = 1$ arises in this way ([W]). For this reason, we express the manifold thus obtained by $Y = (\mathfrak{g}, \alpha_r)$. Since this notation depends on the numbering of the simple roots, we fix it as in Table 1. In this table, the notation "◎ r" implies that the Kähler C-space $(\mathfrak{g}, \alpha_r)$ is an *irreducible Hermitian symmetric space of compact type*. In Table 2, we collect all Kähler C-spaces with $b_2 = 1$ (up to isomorphisms) together with some numerical invariants of them such as $\dim Y$, $k(Y)$ and $m(Y)$ (cf. (0.1) and (0.2)).

(1.3) We denote by $\mathcal{O}_Y(1)$ the homogeneous vector bundle on $Y = G/U = (\mathfrak{g}, \alpha_r)$ induced by the irreducible representation of U with the lowest weight $-\lambda_r$. It is known that this is a line bundle and generates $Pic(Y)$. Further, we have the following :

(1.4) **Lemma.**([ST]) *For any positive integer* a , *the line bundle* $\mathcal{O}_Y(a) = \mathcal{O}_Y(1)^{\otimes a}$ *is normally generated, that is, the multiplication map*

$$Sym^b(H^0(\mathcal{O}_Y(a))) \longrightarrow H^0(\mathcal{O}_Y(ab)) ,$$

is surjective for any positive integer b .

Let

(1.5) $$S = \bigoplus_{a\geq 0} S^a \ , \qquad S^a := H^0(\mathcal{O}_Y(a)) \ ,$$

be the homogeneous coordinate ring of Y. We can show the following by means of the Main Theorem in [ST] :

(1.6) **Theorem.** *The Koszul sequence,*

$$\wedge^2 S^a \otimes S^{b-a} \longrightarrow S^a \otimes S^b \longrightarrow S^{a+b} \longrightarrow 0 \ ,$$

is exact provided $0 < a < b$ *and* $b \geq m(Y)$.

We call the projective embedding

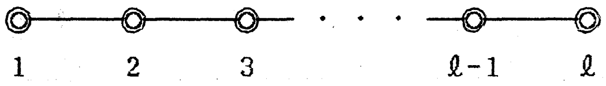
$$v_a : Y \longrightarrow \mathbb{P}(S^a) \ , \quad a > 0 \ ,$$

given by the complete linear system $|\mathcal{O}_Y(a)|$, the *a-th Veronese embedding* of Y . The following is a direct consequence of (1.6).

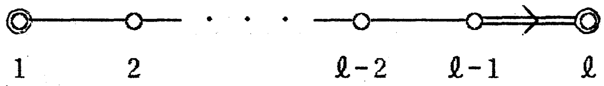
(1.7) **Theorem.** *The homogeneous ideal of the a-th Veronese image* $v_a(Y)$ *is generated in degree* $\leq [m(Y)/a]$. *In particular, it is generated by quadrics if* $3a > m(Y)$.

Table 1.

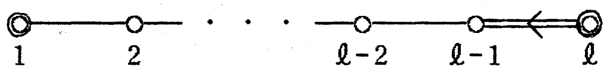
A_ℓ :



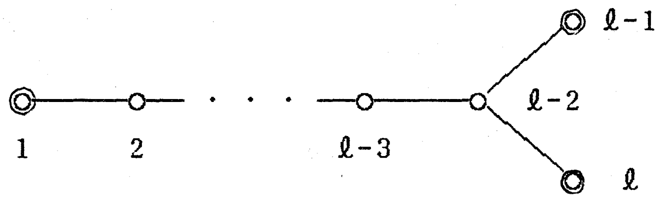
B_ℓ :



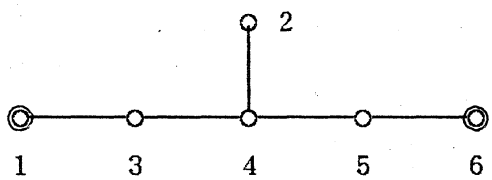
C_ℓ :



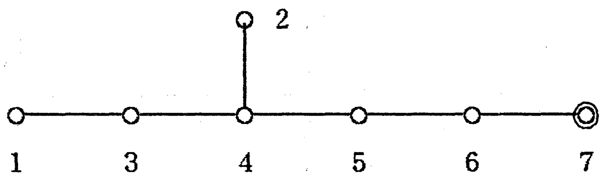
D_ℓ :



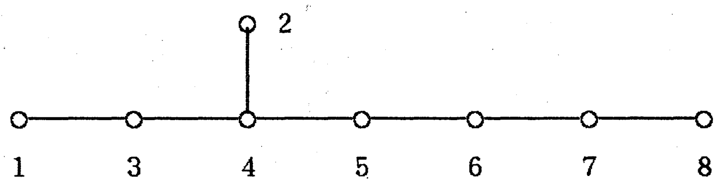
E_6 :



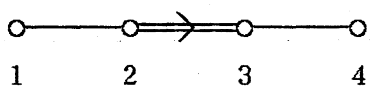
E_7 :



E_8 :



F_4 :



G_2 :

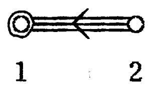


Table 2.

$\mathfrak{g}$	r	N = dim Y	k(Y)	m(Y)
A_ℓ $(\ell \geq 1)$	$1 \leq r \leq \ell+1-r$	$r(\ell+1-r)$	$\ell + 1$	2
B_ℓ $(\ell \geq 2)$	$1 \leq r \leq \ell - 1$	$2r(\ell-r)+r(r+1)/2$	$2\ell - r$	2 $(r = 1)$ 3 $(r \geq 2)$
C_ℓ $(\ell \geq 3)$	$2 \leq r \leq \ell$	$2r(\ell-r)+r(r+1)/2$	$2\ell-r+1$	3 $(r < \ell)$ 2 $(r = \ell)$
D_ℓ $(\ell \geq 3)$	$1 \leq r \leq \ell-2$	$2r(\ell-r)+r(r-1)/2$	$2\ell-r-1$	2 $(r = 1)$ 3 $(r \geq 2)$
	ℓ	$\ell(\ell - 1)/2$	$2\ell - 2$	2
E_6	1	16	12	2
	2	21	11	3
	3	25	9	3
	4	29	7	4
E_7	1	33	17	3
	2	42	14	3
	3	47	11	4
	4	53	8	5
	5	50	10	4
	6	42	13	3
	7	27	18	2
E_8	1	78	23	3
	2	92	17	4
	3	98	13	5
	4	106	9	7
	5	104	11	6
	6	97	14	5
	7	83	19	4
	8	57	29	3
F_4	1	15	8	3
	2	20	5	4
	3	20	7	4
	4	15	11	3
G_2	2	5	3	3

2. Vanishing theorems for $H^q(\Omega_Y^p(a))$. To discuss the vanishing of $H^q(\Omega_Y^p(a))$, we use the *generalized Borel-Weil theorem* ([Kos, p. 317]) which treats the cohomology of a homogeneous vector bundle on $Y = G/U$ induced by an *irreducible* representation of U .

It is known that the subalgebra

$$(2.1) \qquad \mathfrak{n}^+ = \sum_{\alpha\in\Phi(\mathfrak{n}^+)} \mathfrak{g}_\alpha , \qquad \Phi(\mathfrak{n}^+) = \{\alpha \in \Phi \mid n_r(\alpha) > 0 \} ,$$

of $\mathfrak{g}$ is invariant by $\mathrm{Ad}(U)$ and induces the cotangent bundle Ω_Y^1 . Thus, $\wedge^p\mathfrak{n}^+$ induces Ω_Y^p . This tells us the impotance to study the U-module structure of $\wedge^p\mathfrak{n}^+$. In case of irreducible Hermitian symmetric spaces of compact type, Kostant [Kos] showed that $\wedge^p\mathfrak{n}^+$ is a completely reducible U-module and wrote down its irreducible decomposition in the language of the Weyl group. Thus, in this case, we already have the essential tools. In fact, by means of this decomposition and the generalized Borel-Weil theorem, Kimura [Ki.1] obtained the complete vanishing theorem for $H^q(\Omega_Y^p(a))$ in case $Y = Q^N$, (E_6, α_1) and (E_7, α_7) . On the other hand, if Y is not symmetric, the U-module $\wedge^p\mathfrak{n}^+$ is not completely reducible. We introduced in [Kon.1] a filtration on $\wedge^p\mathfrak{n}^+$ such that the quotients $G^i(\wedge^p\mathfrak{n}^+)$ are completely reducible. Though its irreducible decomposition still remains open, we can show the following two theorems.

(2.2) **Theorem.** *Let* $Y = (\mathfrak{g}, \alpha_r)$ *be a Kähler C-space with* $b_2(Y) = 1$. *If positive integers* a *and* p *satisfy the following conditions, then* $H^q(\Omega_Y^p(a))$ *vanishes for any* $q \geq 1$.

(1) α_r *is a long root :* $a \geq p$.

(2) $Y = (C_l, \alpha_r)$, (F_4, α_4) : $a \geq \min(p + 1, 2p - 1)$.

(3) $Y = (F_4, \alpha_3)$: $a \geq \min(p + 3, 2p - 1)$.

(2.3) **Theorem.** *Let* Y *be as in* (2.2). *Set* $\dim Y = N$ *and assume that positive integers* p , q *and an integer* a *satisfy the following conditions. Then* $H^q(\Omega_Y^p(a)) = 0$.

(1) $Y \neq (F_4, \alpha_3)$: (i) $p + 1 - k(Y) \leq a \leq p$,

(ii) $q \geq N + p + 1 - k(Y) - a$.

(2) $Y = (F_4, \alpha_3)$:

(i) $\min(p + 4, 2p) - k(Y) \leq a \leq \min(p + 3, 2p - 1)$,

(ii) $q \geq N + \min(p + 4, 2p) - k(Y) - a$.

We note that we can make (2.3) sharper if we consider Y individually. Further, we have some concrete informations about the vanishing of $H^q(\Omega_Y^p(a))$ for small p as well. For the detail, see [Kon.2].

3. Jacobian rings and IVHS.

(3.1) Let X be a smooth hypersurface defined by a section $f \in S^d$. Then the $(N-1)$-th cohomology $H^{N-1}(X)$ admits the *Hodge decomposition*. Further, we can define the *primitive part* by means of the ample class $\omega = c_1(\mathcal{O}_X(d)) \in H^1(\Omega_X^1)$. In this way, we get a *polarized Hodge structure* $\{H_{\mathbb{Z}}, H^{p,q}, Q\}$ of weight $(N - 1)$, where

(PHS.1) $H_{\mathbb{Z}} = H^{N-1}(X, \mathbb{Z}) \cap H^{N-1}_{prim}(X, \mathbb{Q})$,

(PHS.2) $H^{p,q} = H^{p,q}(X) \cap H^{N-1}_{prim}(X, \mathbb{C})$, (PHS.3) $Q : H_{\mathbb{Z}} \times H_{\mathbb{Z}} \longrightarrow H_{\mathbb{Z}}$, the cup-product.

Let

(3.1.1) $T = \{\theta \in H^1(T_X) \mid \theta\cdot\omega = 0\}$

be the Kuranishi space of deformations of the polarized manifold (X, ω) . Then, as is well known, the differential of the period map is identified with the so-called *infinitesimal period map*

(3.1.2) $\nu : T \longrightarrow \bigoplus_{p+q=N-1} \mathrm{Hom}_{\mathbb{C}}(H^{p,q}, H^{p-1,q+1})$,

induced by the contruction $T_X \otimes \Omega_X^p \longrightarrow \Omega_X^{p-1}$. The data $\{H_{\mathbb{Z}}, H^{p,q}, Q, T, \nu\}$ is the *infinitesimal variation of Hodge structure* (abbreviated IVHS) of X . Thanks to the linear map ν , IVHS has rich "algebraic" structure. In fact, we can interprit the first piece of the algebraic part $\{H^{p,q}, T, \nu\}$ of IVHS in the language of the *Jacobian ring* of X .

(3.2) Before recalling the definition of the Jacobian ring due to Green [G], we need some preparations. Put $L = \mathcal{O}_Y(d)$ and let Σ_L be the sheaf of first order differential operators on sections of L (the *first prolongation bundle* in the terminology of [G]). There are two fundamental exact sequences involving Σ_L . The first is induced by the symbol map $\Sigma_L \longrightarrow T_Y$:

(3.2.1) $0 \longrightarrow \mathcal{O}_Y \longrightarrow \Sigma_L \longrightarrow T_Y \longrightarrow 0$.

The contruction with the *1-jet extension* $j(f)$ of f yields the second :

(3.2.2) $0 \longrightarrow T_Y(-\log X) \longrightarrow \Sigma_L \xrightarrow{\cdot j(f)} L \longrightarrow 0$,

where $T_Y(-\log X)$ is the subsheaf of T_Y consisting of derivations which sends the ideal sheaf of X into itself.

(3.3) Let a be a non-negative integer. Tensoring (3.2.2) with $\mathcal{O}_Y(a-d)$ and taking cohomology, we get a natural map

$$\rho_a : H^0(\Sigma_L(a-d)) \xrightarrow{\cdot j(f)} S^a = H^0(\mathcal{O}_Y(a)) .$$

Set $J_X^a = \mathrm{Ker}(\rho_a)$ and $R_X^a = \mathrm{Coker}(\rho_a)$. We call

(3.3.1) $$J = J_X = \bigoplus_{a\geq 0} J_X^a ,$$

and

(3.3.2) $$R = R_X = \bigoplus_{a\geq 0} R_X^a ,$$

the *Jacobian ideal* and the *Jacobian ring* of X , respectively.

We can show the following :

(3.4) **Lemma.** *If Y is not a complex projective space, then the Jacobian ideal J_X of a smooth hypersurface X of degree d is generated in degree d . In particular, $R_X^a = S^a$ if $a < d$.*

(3.5) By means of the dual sequence of (3.2.2), we can construct the following exact Koszul sequence :

$$(\mathrm{KS})_p: \quad 0 \longrightarrow L^{-p} \longrightarrow \Sigma_L^* \otimes L^{1-p} \longrightarrow \cdots \longrightarrow \wedge^i\Sigma_L^* \otimes L^{i-p} \longrightarrow \cdots$$
$$\cdots \longrightarrow \wedge^{p-1}\Sigma_L^* \otimes L^{-1} \longrightarrow \wedge^p\Sigma_L^* \longrightarrow \Omega_Y^p(\log X) \longrightarrow 0 .$$

Then $(\mathrm{KS})_{p+1}$ and $(\mathrm{KS})_{N+1}$ yield the exact sequence

$$0 \longrightarrow \Omega_Y^{p+1}(\log X) \longrightarrow \wedge^{p+2}\Sigma_L^* \otimes L \longrightarrow \cdots$$
$$\cdots \longrightarrow \wedge^N\Sigma_L^* \otimes L^{N-p-1} \longrightarrow \wedge^{N+1}\Sigma_L^* \otimes L^{N-p} \longrightarrow 0 .$$

Note that we have $\wedge^{N+1}\Sigma_L^* \simeq K_Y$ by (3.2.1) and, there is a canonical isomorphism $\wedge^i\Sigma_L^* \simeq \wedge^{N+1-i}\Sigma_L \otimes \wedge^{N+1}\Sigma_L^*$. Put

(3.5.1) $t(p) = (N-p)d - k(Y)$.

Then the above sequence yields the following (cf. [G, p. 137]):

(3.6) **Lemma.** *Assume that the following conditions are satisfied.*

(1) $H^{N-p-s}(\wedge^{p+1+s}\Sigma_L^*(sd)) = 0$ *for* $1 \leq s \leq N-p-1$.

(2) $H^{s-1}(\wedge^{N+1-s}\Sigma_L^*((N-p-s)d)) = 0$ *for* $2 \leq s \leq N-p-1$.

Then $R^{t(p)} \simeq H^{N-1-p}(\Omega_Y^{p+1}(\log X))$.

The conditions in (3.6) can be simplified if we use

(3.6.1) $0 \longrightarrow \Omega_Y^p \longrightarrow \wedge^p\Sigma_L^* \longrightarrow \Omega_Y^{p-1} \longrightarrow 0$,

derived from the dual of (3.2.1). Further, the Poincaré residue sequence

(3.6.2) $0 \longrightarrow \Omega_Y^{p+1} \longrightarrow \Omega_Y^{p+1}(\log X) \longrightarrow \Omega_X^p \longrightarrow 0$,

will connect $R^{t(p)}$ with $H_{prim}^{N-1-p}(\Omega_X^p)$ passing through $H^{N-1-p}(\Omega_Y^{p+1}(\log X))$. Summing up, we would have

(3.7) **Proposition.** *If the following conditions are satisfied,* *then* $R^{t(p)} \simeq H^{N-1-p}_{prim}(\Omega^p_X)$.

$(\mathrm{HC}\text{-}1)_p$: $2p \neq N-2$.

$(\mathrm{HC}\text{-}2)_p$: $H^{N-t}(\Omega^t_Y(-(N-p-t)d)) = 0$ *for* $1 \leq t \leq N-1-p$.

$(\mathrm{HC}\text{-}3)_p$: $H^{N-t}(\Omega^t_Y(-(N-p-1-t)d)) = 0$ *for* $1 \leq t \leq N-2-p$.

$(\mathrm{HC}\text{-}4)_p$: $H^{N-t+1}(\Omega^t_Y(-(N-p-t)d)) = 0$ *for* $2 \leq t \leq N-1-p$.

Further, we have

(3.8) **Lemma.** *Assume that* Y *is not a projective space.*

(1) $R^d \simeq H^1(T_Y(-\log X)) \simeq T$ *except in the following cases* :

(*a*) $Y = Q^N$ *and* $d = 2$, (*b*) $Y = Q^3$ *and* $d = 3$,

(*c*) $Y = (C_l, \alpha_2)$, (F_4, α_4) , *and* $d = 1$.

(2) *There is a commutative diagram*

$$\begin{array}{ccc}
R^d & \longrightarrow & \mathrm{Hom}(R^{t(p)}, R^{t(p-1)}) \\
\downarrow & & \downarrow \\
H^1(T_Y(-\log X)) & \longrightarrow & \mathrm{Hom}(H^{N-1-p}(\Omega^{p+1}_Y(\log X)), H^{N-p}(\Omega^p_Y(\log X))) \\
\downarrow & & \downarrow \\
T & \longrightarrow & \mathrm{Hom}(H^{p,N-1-p}, H^{p-1,N-p}) .
\end{array}$$

By (3.7), (3.8) and the vanishing theorems for $H^q(\Omega^p_Y(a))$, we get the following :

(3.9) **Theorem.** *Let* Y *be an* N*-dimensional Kähler C-space with* $b_2(Y) = 1$ *which is not a complex projective space. Let* X *be a smooth hypersurface of degree* d , $d > m(Y)$, *and* R *be the Jacobian ring of* X . *Then, for the positive integer* p_0

satisfying $t(p_0) \geq 0$ *and* $t(p_0+1) < 0$, *the infinitesimal period map* $T \otimes H^{p,N-1-p} \longrightarrow H^{p-1,N-p}$ *can be identified with the multiplication map*

$$R^d \otimes R^{t(p_0)} \longrightarrow R^{t(p_0-1)}$$

except in the following cases :

(1) (A_4, α_2) , (A_5, α_2) , Q^3 , (D_6, α_6) , (E_6, α_1) : $d = 3$.

(2) (D_5, α_5) : $d = 3, 6$.

(3) Q^N , $N \geq 4$:

(a) $d = 3$ and $3 | N-2i$ for some i , $1 \leq i \leq 5$.

(b) $d = 4, 5$ and $d | N-2i$ for some i , $1 \leq i \leq 3$.

(c) $d \geq 6$ and $d | N-2i$ for some i , $1 \leq i \leq 2$.

(4) (C_l, α_2); $l \geq 4$, (B_l, α_2); $l \geq 3$, (D_l, α_2); $l \geq 4$, (E_6, α_2) , (E_7, α_1) , (E_8, α_8) , (F_4, α_1) : $d | k(Y)-1$, $d | k(Y)-2$.

(5) (B_l, α_3); $l \geq 4$, (B_l, α_4); $l \geq 5$, (C_l, α_3); $l \geq 4$, (C_l, α_4); $l \geq 5$, (D_l, α_3); $l \geq 5$, (D_l, α_4); $l \geq 6$, (E_6, α_4) , (E_7, α_3) , (E_7, α_6) , (E_8, α_1) , (F_4, α_3) : $d | k(Y)-1$.

(6) (C_3, α_2) , (F_4, α_4) : $d | k(Y)-i$ for some i , $1 \leq i \leq 3$.

4. The Duality Theorem and the Symmetrizer Lemma.

Put

(4.1) $\sigma = \sigma(Y, d) = (N + 1)d - 2k(Y)$.

Tensoring $(\mathrm{KS})_{N+1}$ with $\mathcal{O}_Y(\sigma - a - k)$, we obtain the exact Koszul sequence :

$$0 \longrightarrow \mathcal{O}_Y(-a-k) \longrightarrow \Sigma_L^*(d-a-k) \longrightarrow \cdots \longrightarrow \wedge^p\Sigma_L^*(pd-a-k) \longrightarrow \cdots$$
$$\cdots \longrightarrow \wedge^N\Sigma_L^*(Nd-a-k) \longrightarrow \mathcal{O}_Y(\sigma-a) \longrightarrow 0 .$$

Then, following the analoguous steps as in §3, we can show

(4.2) **Duality Theorem.** *Let* Y *be a Kähler C-space with* $b_2(Y) = 1$. *Assume that* Y *is not a complex projective space and* $\dim Y \geq 3$. *Let* R *be the Jacobian ring of a smooth hypersurface of degree* d *of* Y , $d > m(Y)$. *Then, for the integer* σ *in (4.1),* $R^\sigma \simeq \mathbb{C}$. *Consider the natural pairing*

$$(\mathrm{DP})_a : \quad R^a \otimes R^{\sigma-a} \longrightarrow R^\sigma \simeq \mathbb{C} ,$$

for an integer a *with* $0 \leq a \leq d$. *Then* $(\mathrm{DP})_a$ *gives the exact sequence,*

$$(\mathrm{DS})_a : \quad R^{\sigma-a} \longrightarrow (R^a)^* \longrightarrow 0 ,$$

except for the case : $Y = Q^3$ *and* $a = d = 3$. *Further* $(\mathrm{DP})_a$ *gives the exact sequence,*

$$(\mathrm{DI})_a : \quad 0 \longrightarrow R^{\sigma-a} \longrightarrow (R^a)^* ,$$

except in the following cases.

(1) $Y = Q^N$: $(d, a) = (d, d-2), (3, 2), (3, 3), (4, 4)$.

(2) $Y = (A_5, \alpha_2)$, (E_6, α_1) : $(d, a) = (3, 3)$.

(3) $Y = (C_l, \alpha_2)$, (F_4, α_4) : $(d, a) = (d, d-1)$.

(4.3) We recall the notion of the symmetrizer introduced by Donagi [D]. Let V_1 , V_2 and V_3 be vector spaces and suppose that we are given a bilinear map

(4.3.1) $\quad B : V_1 \times V_2 \longrightarrow V_3$.

Consider a homomorphism defined by

$$\begin{array}{ccc} \mathrm{Hom}(V_1, V_2) \simeq V_1^* \otimes V_2 & \longrightarrow & (\wedge^2 V_1)^* \otimes V_3 \\ \Psi & & \Psi \\ P & \longmapsto & B(x_1, P(x_2)) - B(x_2, P(x_1)) , \quad x_1, x_2 \in V_1 . \end{array}$$

If we denote by V_0 the kernel of this map, then we get a new bilinear map

(4.3.2) $\quad B\_ : V_0 \times V_1 \longrightarrow V_2 , \quad (P, x) \longmapsto P(x)$.

We say that $B\_$ is the *symmetrizer* of B .

(4.4) **Symmetrizer Lemma.** *Let Y and R be as in (4.2). Let a and b be positive integers satisfying $a \le b \le d$. Then the Koszul sequence,*

(SL) : $\quad 0 \longrightarrow R^{b-a} \longrightarrow (R^a)^* \otimes R^b \longrightarrow (\wedge^2 R^a)^* \otimes R^{a+b}$,

is exact except in the following cases :

(1) $Y = Q^N$, $N \ge 3$: $b - a = d - 2$, $d = 3$ *and* $b - a = 2$.

(2) $Y = Q^3$: $a = b = d = 3$.

(3) $Y = (C_l, \alpha_2)$; $l \ge 3$, (F_4, α_4) : $b - a = d - 1$.

In other words, the symmetrizer of the multiplication map

$$B_{a,b} : R^a \otimes R^b \longrightarrow R^{a+b}$$

is nothing but the multiplication map

$$B_{b-a,a} : R^{b-a} \otimes R^a \longrightarrow R^b ,$$

except in the above cases.

Sketch of the proof. Dualizing (SL) and connecting the natural maps $R^{\sigma-c} \longrightarrow (R^c)^*$ and $S^{\sigma-c} \longrightarrow R^{\sigma-c}$, we get the commutative diagram

$$\begin{array}{ccccccc}
\wedge^2 R^a \otimes (R^{a+b})^* & \longrightarrow & R^a \otimes (R^b)^* & \longrightarrow & (R^{b-a})^* & \longrightarrow & 0 \\
\uparrow & & \uparrow & & \uparrow & & \\
\wedge^2 R^a \otimes R^{\sigma-(a+b)} & \longrightarrow & R^a \otimes R^{\sigma-b} & \longrightarrow & R^{\sigma-(b-a)} & \longrightarrow & 0 \\
\uparrow & & \uparrow & & \uparrow & & \\
\wedge^2 S^a \otimes S^{\sigma-(a+b)} & \longrightarrow & S^a \otimes S^{\sigma-b} & \longrightarrow & S^{\sigma-(b-a)} & \longrightarrow & 0 .
\end{array}$$

It follows from (1.6) that the bottom row is exact except for the case (*2*). Since, by (3.4), the Jacobian ideal is generated in degree d , it is easy to see that the middle row is exact, too. By diagram chasing, we see that the top row is exact if $R^{\sigma-b} \longrightarrow (R^b)^*$ is surjective and $R^{\sigma-(b-a)} \longrightarrow (R^{b-a})^*$ is bijective. Then, the assertion follows from (4.2). **Q.E.D.**

5. Sketch of the proof of (0.3). We can show the following as in the case of projective hypersurfaces (cf. [PS]).

(5.1) **Proposition.** *Let* Y *be a Kähler C-space with* $b_2(Y) = 1$ *not isomorphic to a complex projective space and* d *an integer such that* $d > m(Y)$.

(*1*) *Two smooth hypersurfaces of degree* d *are isomorphic to each other if and only if they are related by an automorphism of* Y .

(*2*) *If* $U_d \subset \mathbb{P}(S^d)$ *is the Zariski open subset parametrizing all smooth hypersurfaces, then every closed point of* U_d *is a stable point under the action of* $\mathrm{Aut}(Y)$ *and the quotient space* $\mathcal{M}_d = U_d/\mathrm{Aut}(Y)$ *is the coarse moduli space of smooth hypersurfaces* *of degree* d . *Further, there exists an* $\mathrm{Aut}(Y)$*-invariant Zariski open subset* $\mathring{U}_d$ *of* U_d *such that the quptient space* $\mathring{\mathcal{M}}_d = \mathring{U}_d/\mathrm{Aut}(Y)$ *is smooth and there is a family of smooth hypersurfaces over* $\mathring{\mathcal{M}}_d$.

(3) *The tangent space* $T_{[X]}(\mathring{\mathcal{M}}_d)$ *at* $[X] \in \mathring{\mathcal{M}}_d$ *is isomorphic to* R^d_X *and the tangent space along the fibre of* $\mathring{U}_d \longrightarrow \mathring{\mathcal{M}}_d$ *at* X *is isomorphic to* J^d_X . *For two smooth hypersurfaces* X *and* Z *of degree* d , $J^d_X = J^d_Z$ *in* S^d *holds if and only if they are related by an automorphism of* Y .

(4) *The period map* $P : \mathcal{M}_d \longrightarrow \Gamma\backslash\mathcal{D}$ *is well-defined and has a regular value, where* $\mathcal{D}$ *is a Griffith domain and* Γ *the monodromy group.*

(5) *The period map* $\mathring{P} = P|_{\mathring{\mathcal{M}}_d} : \mathring{\mathcal{M}}_d \longrightarrow \Gamma\backslash\mathcal{D}$ *is an immersion and partially compactified as a proper holomorphic map.*

(5.2) Let X and Z be two smooth hypersurfaces in $\mathring{U}_d$ and assume that $\mathring{P}([X]) = \mathring{P}([Z])$. Then they have isomorphic IVHS and, under the hypoteses in (0.3), we have a commutative diagram

$$(\mathrm{SD})_0 : \qquad \begin{array}{ccccc} R^d_X & \otimes & R^{t(p_0)}_X & \longrightarrow & R^{t(p_0-1)}_X \\ \varphi_d \downarrow & & \downarrow \varphi_{t(p_0)} & & \downarrow \varphi_{t(p_0-1)} \\ R^d_Z & \otimes & R^{t(p_0)}_Z & \longrightarrow & R^{t(p_0-1)}_Z \end{array}$$

by (3.9), where the horizontal maps are multiplication and the vertical maps $\varphi_.$'s are unknown isomorphisms. Applying the Symmetrizer Lemma (4.4) successively, we obtain a sequence of commutative diagrams

$$(\mathrm{SD})_i : \quad \begin{array}{ccccc} R_X^{a(i)} & \otimes & R_X^{b(i)} & \longrightarrow & R_X^{a(i)+b(i)} \\ {\scriptstyle \varphi_{a(i)}}\downarrow & & \downarrow{\scriptstyle \varphi_{b(i)}} & & \downarrow{\scriptstyle \varphi_{a(i)+b(i)}} \\ R_Z^{a(i)} & \otimes & R_Z^{b(i)} & \longrightarrow & R_Z^{a(i)+b(i)} \end{array}$$

which terminates at

$$(\mathrm{SD})_{end} : \quad \begin{array}{ccccc} R_X^{g} & \otimes & R_X^{g} & \longrightarrow & R_X^{2g} \\ {\scriptstyle \varphi_{g}}\downarrow & & \downarrow{\scriptstyle \varphi'_{g}} & & \downarrow{\scriptstyle \varphi_{2g}} \\ R_Z^{g} & \otimes & R_Z^{g} & \longrightarrow & R_Z^{2g} \end{array}$$

where $g = \mathrm{G.C.D.}(d,\ k(Y))$. Since we have assumed that $2g < d$, we see from (3.4) that $R_X^g = R_Z^g = S^g$ and $R_X^{2g} = R_Z^{2g} = S^{2g}$. Further, it is easy to see that φ_g and φ'_g are essentially the same. Thus the diagram $(\mathrm{SD})_{end}$ is nothing but

$$\begin{array}{ccccc} S^{g} & \otimes & S^{g} & \longrightarrow & S^{2g} \\ {\scriptstyle \varphi_{g}}\downarrow & & \downarrow{\scriptstyle \varphi_{g}} & & \downarrow{\scriptstyle \varphi_{2g}} \\ S^{g} & \otimes & S^{g} & \longrightarrow & S^{2g} \end{array} .$$

There may be two possibilities : (i) $3g > m(Y)$, (ii) $3g \leq m(Y)$. We treat the first case only, since the latter is quite similar (but slightly complicated). The above diagram yields

$$\begin{array}{ccccc} I^2_{(g)} & \longrightarrow & Sym^2(S^g) & \longrightarrow & S^{2g} \\ \downarrow & & \downarrow{\scriptstyle \varphi_g \otimes \varphi_g} & & \downarrow{\scriptstyle \varphi_{2g}} \\ I^2_{(g)} & \longrightarrow & Sym^2(S^g) & \longrightarrow & S^{2g} \end{array} ,$$

where $I^2_{(g)}$ denotes the degree 2 piece of the defining ideal of the g-th Veronese image of Y. By (1.7) and the above diagram, we see that φ_g comes from an automorphism of Y. Then going back the symmetrizer step one by one, we see that the unknown isomorphism φ_c appearing in $(\mathrm{SD})_i$ comes from φ_g so long as $c < d$ (since $R_X^c = R_Z^c = S^c$). Finally, we would arrive at the commutative diagram,

$$\begin{tikzcd}
 & S^d \arrow[dr] \arrow[ddd, "\tilde{\varphi}_d", "\wr"'] & \\
S^a \otimes S^b \arrow[ur] \arrow[rr] \arrow[d, "\varphi_a \wr \quad \wr \varphi_b"'] & & R_X^d \arrow[d, "\wr\, \varphi_d"] \\
S^a \otimes S^b \arrow[dr] \arrow[rr] & & R_Z^d \\
 & S^d \arrow[ur] &
\end{tikzcd}$$

with φ_a, φ_b and $\tilde{\varphi}_d$ come from φ_g, where a and b are some positive integers such that $a + b = d$. Thus we have $\tilde{\varphi}_d(J_X^d) = J_Z^d$. Since φ_g (and thus $\tilde{\varphi}_d$) comes from an automorphism of Y, we may assume that $J_X^d = J_Z^d$ in S^d. Thus (0.3) follows from (5.1), (3).

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Mathematical Institute
Tôhoku University
Sendai, 980
Japan